

**PFAFFIAN L -ENSEMBLES RELATED TO THE
Z-MEASURES ON PARTITIONS WITH THE JACK
PARAMETERS $\theta = \frac{1}{2}, 2$.**

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ABSTRACT. We construct Pfaffian L -ensembles related to the z -measures on partitions and to the Plancherel measures on partitions with the Jack parameters $\theta = \frac{1}{2}, 2$. The results imply that these measures on partitions lead to Pfaffian point processes, and the correlation kernels of these processes can be expressed in terms of the corresponding L -matrices. We give explicit formulae for these L -matrices.

1. INTRODUCTION

It is known that measures on partitions arising in the context of the representation theory of the infinite symmetric group lead to determinantal point processes. The most well known example is the Plancherel measure on partitions studied in many papers, see, for example, Logan and Shepp [23], Vershik and Kerov [32], Baik, Deift and Johansson [1, 2], Deift [17], Ivanov and Olshanski [19]. The relation with determinantal point processes was established in Borodin, Okounkov, and Olshanski in Ref. [7], and by Johansson in Ref. [20]. To explain this relation let us identify partitions of n with Young diagrams containing n boxes. The set of such Young diagrams will be denoted as \mathbb{Y}_n . Let $\mathbb{Y} = \mathbb{Y}_0 \cup \mathbb{Y}_1 \cup \dots$ be the set of all Young diagrams including the empty diagram \emptyset . Consider the poissonized Plancherel measure $M_{\text{Pl},\eta}$ on \mathbb{Y} obtained by mixing together the Plancherel measures $M_{\text{Pl}}^{(n)}$ on \mathbb{Y}_n , $n = 0, 1, 2, \dots$,

$$M_{\text{Pl},\eta}(\lambda) = e^{-\eta} \frac{\eta^{|\lambda|}}{|\lambda|!} M_{\text{Pl}}^{(|\lambda|)}(\lambda), \quad \lambda \in \mathbb{Y}.$$

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Here $\eta > 0$. Under the correspondence $\lambda \rightarrow \{\lambda_i - i + \frac{1}{2}\}$ the poissonized Plancherel measure $M_{\text{Pl},\eta}$ turns into the determinantal point process on the lattice $\mathbb{Z} + \frac{1}{2}$ whose correlation kernel is the discrete Bessel kernel, see Theorems 1, 2 in Borodin, Okounkov, and Olshanski [7].

The z -measures $M_{z,z'}^{(n)}$ on partitions is another important class of measures leading to determinantal point processes on $\mathbb{Z} + \frac{1}{2}$. These measures are parameterized by two complex parameters z, z' , and were first introduced in Kerov, Olshanski, and Vershik [22] in the context of the harmonic analysis on the infinite symmetric group. The z -measures $M_{z,z'}^{(n)}$ were studied in detail by Borodin and Olshanski [8]-[11], see also related papers by Okounkov [26], Borodin, Olshanski, and Strahov [13]. The relation with determinantal point processes arises in a way similar to that in the case of the Plancherel measure. Namely, consider the mixed z -measures $M_{z,z',\xi}$ with an additional parameter $\xi \in (0, 1)$ obtained by mixing up the z -measures $M_{z,z'}^{(n)}$,

$$M_{z,z',\xi}(\lambda) = (1 - \xi)^{zz'} \frac{(zz')^{|\lambda|} \xi^{|\lambda|}}{|\lambda|!} M_{z,z'}^{(n)}(\lambda),$$

where λ ranges over \mathbb{Y} . Then under the correspondence $\lambda \rightarrow \{\lambda_i - i + \frac{1}{2}\}$ the measure $M_{z,z',\xi}$ turns into a determinantal point process on the lattice $\mathbb{Z} + \frac{1}{2}$. The correlation kernel of this point process can be written in terms of the Gaussian hypergeometric functions, see Borodin and Olshanski [8].

In the examples described above the determinantal point processes have a special remarkable feature: they can be understood as determinantal L -ensembles. To define a determinantal L -ensemble let us introduce a finite set (called a phase space) \mathfrak{X} , and let L be a $|\mathfrak{X}| \times |\mathfrak{X}|$ matrix (called the L -matrix) whose rows and columns are parameterized by points of \mathfrak{X} . If L is positive definite, one can define a random point process on \mathfrak{X} by

$$\text{Prob}\{X\} = \frac{\det L(X|X)}{\det(I + L)},$$

where X is a subset of \mathfrak{X} , and $L(X|X)$ is the symmetric submatrix of L corresponding to X . It is a well-known fact that such L -ensemble is a determinantal point process whose correlation kernel K is given by $K = L(I + L)^{-1}$. The definition of determinantal L -ensembles can be extended to infinite phase spaces \mathfrak{X} provided that the determinant $\det(I + L)$ is well defined. For other properties of determinantal L -ensembles and their applications we refer the reader to Borodin [4], Section 5.

For the poissonized Plancherel measure $M_{\text{Pl},\eta}$, and for the mixed z -measures $M_{z,z',\xi}$ the corresponding L -matrices can be written explicitly in terms of elementary functions only, and these matrices have remarkably simple forms. Moreover, it turns out that the kernels $L(x, y)$ defining the L -matrices are integrable in the sense of Its, Izergin, Korepin, and Slavnov [18]. This leads to an algorithm (based on Riemann-Hilbert problems) to compute explicitly the correlation kernel K , see Borodin [3].

Kerov [21], Borodin and Olshanski [11] have shown that it is natural to consider a deformation $M_{z,z',\theta}^{(n)}$ of $M_{z,z'}^{(n)}$, where $\theta > 0$ is called the parameter of deformation (or the Jack parameter). Such deformations are in many ways similar to log-gas (random-matrix) models with arbitrary $\beta = 2\theta$. As in the theory of log-gas models the value $\beta = 2$ is a distinguished one and leads to determinantal point processes. On the next level of difficulty are the cases $\theta = 2$ or $\theta = 1/2$ ($\beta = 4$ or $\beta = 1$, respectively). In these cases the measures $M_{z,z',\theta}^{(n)}$ lead to Pfaffian point processes, similar to ensembles of Random Matrix Theory of $\beta = 4$ or $\beta = 1$ symmetry types, see Borodin and Strahov [16], Strahov [29]–[31] for the available results in this direction. It turns out that such Pfaffian point processes are of great interest to the harmonic analysis on the infinite symmetric group. The fact that these measures play a role in the harmonic analysis was established by Olshanski [27], and the detailed explanation of this representation-theoretic aspect can be found in Strahov [30].

The aim of this work is to show that the Pfaffian point processes related to the z -measures on partitions with the Jack parameters $\theta = \frac{1}{2}, 2$, and to the Plancherel measures on partitions with the Jack parameters $\theta = \frac{1}{2}, 2$ can be understood as Pfaffian L -ensembles (see Borodin and Rains [14], Borodin and Strahov [15] and Section 2 of the present paper for a definition of Pfaffian L -ensembles). The paper gives explicitly the L -matrices for these ensembles, see Theorem 4.1 and Theorem 4.2.

In the context of the harmonic analysis on the infinite symmetric group the most important problem is to understand the scaling limits of the determinantal and of the Pfaffian point processes defined by the mixing z -measures $M_{z,z',\xi,\theta}$ ($\theta = 1, \frac{1}{2}$, or 2), as $\xi \nearrow 1$. A possible approach to this problem is to study the convergence of the corresponding (determinantal or Pfaffian) L -ensemble to that defined by a limiting bounded operator \mathcal{L} acting in $L^2(\mathbb{R} \setminus \{\emptyset\}) \oplus L^2(\mathbb{R} \setminus \{\emptyset\})$. The limiting operator \mathcal{L} completely characterizes the limiting point process relevant for the harmonic analysis. In particular, the correlation kernel of the limiting process can be expressed in terms of \mathcal{L} . For $\theta = 1$ such a

limit transition was investigated by Borodin [3] (see also Borodin and Olshanski [6] and the references therein).

The results of the present paper (explicit formulae for the L -matrices defining the Pfaffian L -ensembles for $M_{z,z',\xi,\theta=\frac{1}{2}}$ and $M_{z,z',\xi,\theta=2}$) lay a foundation for this approach in the Pfaffian case. The author plans to investigate the transition to such limiting ensembles in a subsequent publication.

The paper is organized as follows. Section 2 contains the definition and some basic properties of Pfaffian L -ensembles. Section 3 contains the definitions of the z -measures and the Plancherel measures on partitions with an arbitrary Jack parameter $\theta > 0$. The main results of the present work are stated in Section 4, see Theorem 4.1 and Theorem 4.2. Section 5 investigates the properties of a special class of Pfaffian L -ensembles, which is relevant for measures on partitions considered in this paper. In Sections 6 and 7 we rewrite the z -measures and Plancherel measures on partitions with the Jack parameters $\theta = \frac{1}{2}, 2$ in terms of suitable (Frobenius-type) coordinates. Then we use formulae obtained in Section 5 to prove Theorem 4.1 and Theorem 4.2.

2. PFAFFIAN L -ENSEMBLES

Let \mathfrak{X} be a countable set. Given \mathfrak{X} let us construct two copies of \mathfrak{X} , and denote them by \mathfrak{X}' and \mathfrak{X}'' . Each point $x \in \mathfrak{X}$ has a prototype $x' \in \mathfrak{X}'$ and another one $x'' \in \mathfrak{X}''$. Let L be a $|\mathfrak{X}| \times |\mathfrak{X}|$ skew symmetric matrix constructed from 2×2 blocks with rows and columns parameterized by elements of $\mathfrak{X}' \times \mathfrak{X}''$. The 2×2 blocks have the form

$$(2.1) \quad L(x, y) = \begin{bmatrix} L(x', y') & L(x', y'') \\ L(x'', y') & L(x'', y'') \end{bmatrix}.$$

Once x, y take values in \mathfrak{X} , the variables x', y' (x'', y'') are the elements of \mathfrak{X}' (\mathfrak{X}'') corresponding to x, y . The matrix L can also be understood as a $2|\mathfrak{X}| \times 2|\mathfrak{X}|$ matrix with rows and columns parameterized by points of \mathfrak{X} .

Let $\text{Conf}(\mathfrak{X})$ be the set of all subsets of \mathfrak{X} and denote by $\text{Conf}(\mathfrak{X})_0 \subset \text{Conf}(\mathfrak{X})$ the set of finite subsets of \mathfrak{X} . To any $X \subset \text{Conf}(\mathfrak{X})_0$ there will correspond a 2×2 block antisymmetric submatrix of L of a finite size. We denote this submatrix by $L(X|X)$. If X consists of m points,

$$X = (x_1, \dots, x_m), \quad X \in \text{Conf}(\mathfrak{X})_0,$$

then the submatrix $L(X|X)$ has the form

$$L(X|X) = \begin{bmatrix} 0 & L(x'_1, x''_1) & \dots & L(x'_1, x'_m) & L(x'_1, x''_m) \\ -L(x'_1, x''_1) & 0 & & L(x''_1, x'_m) & L(x''_1, x''_m) \\ \vdots & & & & \\ -L(x'_1, x'_m) & -L(x'_1, x''_m) & & 0 & L(x'_m, x''_m) \\ -L(x'_1, x''_m) & -L(x''_1, x''_m) & & -L(x'_m, x''_m) & 0 \end{bmatrix}.$$

Denote by $\text{Pf } A$ the Pfaffian of an even dimensional antisymmetric matrix A . In what follows we always assume that the matrix L has the property

$$(2.2) \quad \text{Pf } L(X|X) \geq 0, \quad \forall X \in \text{Conf}(\mathfrak{X})_0,$$

and

$$(2.3) \quad \sum_{X: X \in \text{Conf}(\mathfrak{X})_0} \text{Pf } L(X|X) < \infty.$$

Let J be 2×2 block matrix of format $\mathfrak{X} \times \mathfrak{X}$ with matrix elements

$$(2.4) \quad J(x, y) = \begin{cases} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, & x = y; \\ 0, & \text{otherwise.} \end{cases}$$

Define the expression $\text{Pf } (J + L)$ by the formula

$$(2.5) \quad \text{Pf } (J + L) = \sum_{X: X \in \text{Conf}(\mathfrak{X})_0} \text{Pf } L(X|X).$$

Condition (2.3) ensures that the sum in the righthand side of equation (2.5) is finite. Note that if \mathfrak{X} is a finite set, then L and J are matrices of finite size, and equation (2.5) is the expansion of $\text{Pf } (J + L)$ into a sum of Pfaffians of antisymmetric 2×2 block submatrices $L(X|X)$ of L .

Definition 2.1. A point process on \mathfrak{X} defined by

$$(2.6) \quad \text{Prob}_L(X) = \frac{\text{Pf } L(X|X)}{\text{Pf } (J + L)}, \quad \forall X \in \text{Conf}(\mathfrak{X})_0,$$

is called the Pfaffian L -ensemble.

The fact that $\sum_{X \in \text{Conf}(\mathfrak{X})_0} \text{Prob}_L(X) = 1$ follows from equation (2.5).

By correlation functions $\varrho(X)$ for the Pfaffian L -ensembles we mean the probabilities that random configurations include fixed sets X , namely

$$\varrho(X) = \sum_{Y: Y \in \text{Conf}(\mathfrak{X})_0, Y \supseteq X} \text{Prob}_L(Y).$$

The striking property of the Pfaffian L -ensembles is that the correlation function $\varrho(X)$ is given by a Pfaffian,

$$(2.7) \quad \varrho(X) = \text{Pf} [K(x_i, x_j)]_{i,j=1}^m, \quad X = (x_1, \dots, x_m) \in \text{Conf}(\mathfrak{X})_0.$$

Here the matrix K is a $|\mathfrak{X}| \times |\mathfrak{X}|$ skew symmetric matrix made from 2×2 blocks with rows and columns parameterized by elements of $\mathfrak{X}' \times \mathfrak{X}''$, and defined in terms of L by the expression

$$(2.8) \quad K = J + (J + L)^{-1}.$$

The Pfaffian expression for m -point correlation functions reflects the fact that the Pfaffian L -ensembles is a special class of Pfaffian point processes.

3. THE z -MEASURES ON PARTITIONS WITH THE GENERAL PARAMETER $\theta > 0$

We use Macdonald [24] as a basic reference for the notations related to integer partitions and to symmetric functions. In particular, every decomposition

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l) : n = \lambda_1 + \lambda_2 + \dots + \lambda_l,$$

where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_l$ are positive integers, is called an integer partition. We identify integer partitions with the corresponding Young diagrams, and denote the set of all Young diagrams by \mathbb{Y} . The set of Young diagrams with n boxes is denoted by \mathbb{Y}_n . Thus

$$\mathbb{Y} = \bigcup_{n=0}^{\infty} \mathbb{Y}_n.$$

Following Borodin and Olshanski [11], Section 1, let $M_{z,z',\theta}^{(n)}$ be a complex measure on \mathbb{Y}_n defined by

$$(3.1) \quad M_{z,z',\theta}^{(n)} = \frac{n!(z)_{\lambda,\theta}(z')_{\lambda,\theta}}{(t)_n H(\lambda, \theta) H'(\lambda, \theta)},$$

where $n = 1, 2, \dots$, and where we use the following notation

- $z, z' \in \mathbb{C}$ and $\theta > 0$ are parameters, the parameter t is defined by

$$t = \frac{zz'}{\theta}.$$

- $(t)_n$ stands for the Pochhammer symbol,

$$(t)_n = t(t+1) \dots (t+n-1) = \frac{\Gamma(t+n)}{\Gamma(t)}.$$

- $(z)_{\lambda, \theta}$ is a multidimensional analogue of the Pochhammer symbol defined by

$$(z)_{\lambda, \theta} = \prod_{(i,j) \in \lambda} (z + (j-1) - (i-1)\theta) = \prod_{i=1}^{l(\lambda)} (z - (i-1)\theta)_{\lambda_i}.$$

Here $(i, j) \in \lambda$ stands for the box in the i th row and the j th column of the Young diagram λ , and we denote by $l(\lambda)$ the number of nonempty rows in the Young diagram λ .

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$$H(\lambda, \theta) = \prod_{(i,j) \in \lambda} ((\lambda_i - j) + (\lambda'_j - i)\theta + 1),$$

$$H'(\lambda, \theta) = \prod_{(i,j) \in \lambda} ((\lambda_i - j) + (\lambda'_j - i)\theta + \theta),$$

where λ' denotes the transposed diagram.

Proposition 3.1. *The following symmetry relations hold true*

$$H(\lambda, \theta) = \theta^{|\lambda|} H'(\lambda', \frac{1}{\theta}), \quad (z)_{\lambda, \theta} = (-\theta)^{|\lambda|} \left(-\frac{z}{\theta}\right)_{\lambda', \frac{1}{\theta}}.$$

Here $|\lambda|$ stands for the number of boxes in the diagram λ .

Proof. These relations follow immediately from definitions of $H(\lambda, \theta)$ and $(z)_{\lambda, \theta}$. \square

Proposition 3.2. *We have*

$$M_{z, z', \theta}^{(n)}(\lambda) = M_{-z/\theta, -z'/\theta, 1/\theta}^{(n)}(\lambda').$$

Proof. Use definition of $M_{z, z', \theta}^{(n)}(\lambda)$, equation (3.1), and apply Proposition 3.1. \square

Proposition 3.3. *We have*

$$\sum_{\lambda \in \mathbb{Y}_n} M_{z, z', \theta}^{(n)}(\lambda) = 1.$$

Proof. See Kerov [21], Borodin and Olshanski [11, 5]. \square

Proposition 3.4. *If parameters z, z' satisfy one of the three conditions listed below, then the measure $M_{z, z', \theta}^{(n)}$ defined by expression (3.1) is a probability measure on Y_n . The conditions are as follows.*

- *Principal series:* either $z \in \mathbb{C} \setminus (\mathbb{Z}_{\leq 0} + \mathbb{Z}_{\geq 0}\theta)$ and $z' = \bar{z}$.
- *The complementary series:* the parameter θ is a rational number, and both z, z' are real numbers lying in one of the intervals between two consecutive numbers from the lattice $\mathbb{Z} + \mathbb{Z}\theta$.

- *The degenerate series:* z, z' satisfy one of the following conditions
 - (1) $(z = m\theta, z' > (m-1)\theta)$ or $(z' = m\theta, z > (m-1)\theta)$;
 - (2) $(z = -m, z' < -m+1)$ or $(z' = -m, z < m-1)$.

Proof. See Propositions 1.2, 1.3 in Borodin and Olshanski [11]. \square

In what follows we fix two complex parameters z, z' such that the conditions in Proposition 3.4 are satisfied, and $M_{z,z',\theta}^{(n)}$ is a probability measure on \mathbb{Y}_n .

It is convenient to mix all measures $M_{z,z',\theta}^{(n)}$, and to define a new measure $M_{z,z',\xi,\theta}$ on $\mathbb{Y} = \mathbb{Y}_0 \cup \mathbb{Y}_1 \cup \dots$. Namely, let $\xi \in (0, 1)$ be an additional parameter, and set

$$(3.2) \quad M_{z,z',\xi,\theta}(\lambda) = (1 - \xi)^t \xi^{|\lambda|} \frac{(z)_{\lambda,\theta} (z')_{\lambda,\theta}}{H(\lambda, \theta) H'(\lambda, \theta)}.$$

Proposition 3.5. *We have*

$$\sum_{\lambda \in \mathbb{Y}} M_{z,z',\xi,\theta}(\lambda) = 1.$$

Proof. Follows immediately from Proposition 3.3. \square

Thus $M_{z,z',\xi,\theta}(\lambda)$ is a probability measure on \mathbb{Y} . We will refer to $M_{z,z',\xi,\theta}(\lambda)$ as the mixed z -measure with the deformation (Jack) parameter θ .

When both z, z' go to infinity, expression (3.1) has a limit

$$(3.3) \quad M_{\text{Pl},\theta}^{(n)}(\lambda) = \frac{n! \theta^n}{H(\lambda, \theta) H'(\lambda, \theta)}$$

called the Plancherel measure on \mathbb{Y}_n with general $\theta > 0$. Proposition 3.2 implies that

$$(3.4) \quad M_{\text{Pl},\theta}^{(n)}(\lambda) = M_{\text{Pl},\frac{1}{\theta}}^{(n)}(\lambda').$$

Instead of (3.3), sometimes it is more convenient to consider the Poissonized Plancherel measure with general $\theta > 0$,

$$(3.5) \quad M_{\text{Pl},\eta,\theta}(\lambda) = e^{-\eta} (\eta)^{|\lambda|} \frac{\theta^{|\lambda|}}{H(\lambda, \theta) H'(\lambda, \theta)},$$

where $\eta > 0$. Clearly, $M_{\text{Pl},\eta,\theta}(\lambda)$ is a probability measure on the set \mathbb{Y} .

Remark 3.6. (1) Statistics of the Plancherel measure with the general Jack parameter $\theta > 0$ is discussed in Matsumoto [25]. Matsumoto [25] compares limiting distributions of rows of random partitions with distributions of certain random variables from a traceless Gaussian β -ensemble.

(2) When $\theta = 1$ the poissonized Plancherel measure $M_{\text{Pl},\eta,\theta}(\lambda)$, and

the mixed z -measure $M_{z,z',\xi,\theta}$ lead to dereminantal processes on $\mathbb{Z} + \frac{1}{2}$, see Borodin, Okounkov and Olshanski [7], Johansson [20], Borodin and Olshanski [8].

(3) The poissonized Plancherel measures and certain analogues of the mixed z -measures on the strict partitions are considered in Petrov [28]. Petrov [28] shows that such measures lead to determinantal processes as well.

(4) For $\theta = \frac{1}{2}$ or $\theta = 2$ the poissonized Plancherel measure $M_{\text{Pl},\eta,\theta}(\lambda)$, and the mixed z -measure $M_{z,z',\xi,\theta}$ lead to Pfaffian point processes on $\mathbb{Z} + \frac{1}{2}$, see Strahov [29, 31], and the references therein.

4. MAIN RESULTS

4.1. L -matrices. Set $\mathfrak{X} = \mathbb{Z} + \frac{1}{2}$, $\mathfrak{X}_+ = \mathbb{Z}_{\geq 0} + \frac{1}{2}$, and $\mathfrak{X}_- = \mathbb{Z}_{\leq 0} - \frac{1}{2}$. Introduce the parity on the sets $\mathfrak{X}_+ = \mathbb{Z}_{\geq 0} + \frac{1}{2}$ and $\mathfrak{X}_- = \mathbb{Z}_{\leq 0} - \frac{1}{2}$ referring to $\frac{1}{2}$ and $-\frac{1}{2}$ as to even elements.

According to the decomposition of the set $\mathbb{Z} + \frac{1}{2}$,

$$\mathbb{Z} + \frac{1}{2} = \mathbb{Z}_{\leq 0} - \frac{1}{2} \sqcup \frac{1}{2} \sqcup \mathbb{Z}_{\geq 0} + \frac{3}{2},$$

we write the matrix L in the block form:

$$(4.1) \quad L = \begin{bmatrix} L_{--} & L_{-0} & L_{-+} \\ L_{0-} & L_{00} & L_{0+} \\ L_{+-} & L_{+0} & L_{++} \end{bmatrix}.$$

We are interested in the matrices L defined by

$$(4.2) \quad L = \begin{bmatrix} E & A & B \\ -A^T & 0 & 0 \\ -B^T & 0 & 0 \end{bmatrix}$$

As usual, here E , A , B are the matrices with 2×2 block elements. Specifically,

$$(4.3) \quad E(x, y) = \begin{bmatrix} \epsilon(x, y) & 0 \\ 0 & 0 \end{bmatrix}, \quad x, y \in \mathbb{Z}_{\leq 0} - \frac{1}{2},$$

$$(4.4) \quad A(x, y) = \begin{bmatrix} \epsilon(x, y) & 0 \\ 0 & \frac{h(x)h(y)}{x-y} \end{bmatrix}, \quad x \in \mathbb{Z}_{\leq 0} - \frac{1}{2}, \quad y = \frac{1}{2},$$

$$(4.5) \quad B(x, y) = \begin{bmatrix} 0 & 0 \\ \frac{h(x)h(y)}{x-y} & \frac{h(x)h(y-\frac{1}{2})}{x-y+\frac{1}{2}} \end{bmatrix}, \quad x \in \mathbb{Z}_{\leq 0} - \frac{1}{2}, \quad y \in \mathbb{Z}_{\geq 0} + \frac{3}{2}.$$

The two-point function $\epsilon(x, y)$ in equations above is antisymmetric, $\epsilon(x, y) = -\epsilon(y, x)$. When $x < y$,

$$(4.6) \quad \epsilon(x, y) = \begin{cases} 1, & x - \text{odd}, y - \text{even}, \\ 0, & \text{otherwise.} \end{cases}$$

The function h is nonnegative on $\mathbb{Z} + \frac{1}{2}$.

4.2. Measures on partitions with the Jack parameter $\theta = 2$ as Pfaffian L -ensembles. We define an embedding $\lambda \rightarrow X$ of the set \mathbb{Y} of Young diagrams into the set $\text{Conf}(\mathbb{Z} + \frac{1}{2})_0$ of finite configurations in $\mathbb{Z} + \frac{1}{2}$ as follows. Let λ be a Young diagram. Given $\lambda = (\lambda_1, \dots, \lambda_l)$ we denote by $\lambda \sqcup \lambda$ another Young diagram defined by

$$\lambda \sqcup \lambda = (\lambda_1, \lambda_1, \lambda_2, \lambda_2, \dots, \lambda_l, \lambda_l).$$

Denote by $(P_1, \dots, P_D | Q_1, \dots, Q_D)$ the usual Frobenius coordinates of $\lambda \sqcup \lambda$ (Macdonald [24], §I.1). Thus P_i is the number of squares in the i th row to the right of the diagonal of $\lambda \sqcup \lambda$, Q_i is the number of squares in the i th column below the diagonal of $\lambda \sqcup \lambda$, and $i = 1, \dots, D$. Here D is the number of boxes on the diagonal of $\lambda \sqcup \lambda$. Given λ consider the point configuration $X = X_- \sqcup X_+$ on $\mathbb{Z} + \frac{1}{2}$ defined in terms of the Frobenius coordinates $(P_1, \dots, P_D | Q_1, \dots, Q_D)$ as follows. If D is even, then we set

$$(4.7) \quad X_+ = \left(P_{D-1} + \frac{1}{2}, P_{D-3} + \frac{1}{2}, \dots, P_1 + \frac{1}{2} \right).$$

If D is odd, then we set

$$(4.8) \quad X_- = \left(\frac{1}{2}, P_{D-2} + \frac{1}{2}, P_{D-4} + \frac{1}{2}, \dots, P_1 + \frac{1}{2} \right).$$

(Observe that if D is odd, then $P_D = 0$.) In both cases (when D is either even or odd) we define X_- as

$$(4.9) \quad X_- = \left(-Q_1 - \frac{1}{2}, -Q_2 - \frac{1}{2}, \dots, -Q_D - \frac{1}{2} \right).$$

Equations (4.7)-(4.9) define the embedding $\lambda \rightarrow X$ of \mathbb{Y} into $\text{Conf}_0(\mathbb{Z} + \frac{1}{2})$. Under this embedding any probability measure M on \mathbb{Y} turns into a probability measure on $\text{Conf}(\mathbb{Z} + \frac{1}{2})_0$. (Assume that $X \in \text{Conf}_0(\mathbb{Z} + \frac{1}{2})$, and assume that there is no a Young diagram λ such that X is representable in terms of Frobenius coordinates of $\lambda \sqcup \lambda$ by equations (4.7)-(4.9). Then we agree that the probability of X is zero). Therefore we get a point process on $\mathbb{Z} + \frac{1}{2}$. We will denote by \underline{M} the point process obtained in this way from a probability measure M on \mathbb{Y} .

Let us introduce the following notation. For any complex a and a nonnegative integer n we set

$$[a]_n = \begin{cases} (a+1)(a+3)\dots(a+n-1), & n \text{ is even,} \\ a(a+2)\dots(a+n-1), & n \text{ is odd;} \\ 1, & n = 0. \end{cases}$$

Theorem 4.1. (A) The point process $\underline{M}_{z,z',\xi,\theta=2}$ is the Pfaffian L -ensemble in the sense of Section 2. The corresponding L -matrix is defined by equations (4.1)-(4.6) with

$$(4.10) \quad h(x) = \begin{cases} \frac{[z+1]_{x-\frac{1}{2}}[z'+1]_{x-\frac{1}{2}}}{\Gamma(x+\frac{1}{2})} \xi^{\frac{x}{2}}, & x \in \mathbb{Z}_{\geq 0} + \frac{1}{2}, \\ \frac{[-z]_{-x-\frac{1}{2}}[-z']_{-x-\frac{1}{2}}}{\Gamma(-x+\frac{1}{2})} \xi^{-\frac{x}{2}}, & x \in \mathbb{Z}_{\leq 0} - \frac{1}{2}. \end{cases}$$

Moreover, we have

$$\text{Pf}(J+L) = (1-\xi)^{-\frac{zz'}{2}}.$$

(B) The point process $\underline{M}_{\text{Pl},\eta,\theta=2}$ is the Pfaffian L -ensemble in the sense of Section 2. The corresponding L -matrix is defined by equations (4.1)-(4.6) with

$$(4.11) \quad h(x) = \frac{(2\eta)^{\frac{1}{2}(|x|+\frac{1}{2})}}{\Gamma(|x|+\frac{1}{2})}, \quad x \in \mathbb{Z} + \frac{1}{2}.$$

Moreover, we have

$$\text{Pf}(J+L) = e^\eta.$$

4.3. Measures on partitions with the Jack parameter $\theta = \frac{1}{2}$ as Pfaffian L -ensembles. In this case we define an embedding $\lambda \rightarrow X'$ of the set \mathbb{Y} of Young diagrams into the set $\text{Conf}(\mathbb{Z} + \frac{1}{2})_0$ of finite configurations in $\mathbb{Z} + \frac{1}{2}$ in a slightly different way. Let λ be a Young diagram. Denote by $(P'_1, \dots, P'_D | Q'_1, \dots, Q'_D)$ the usual Frobenius coordinates of $\lambda' \sqcup \lambda$. Given λ consider the point configuration $X' = X'_- \sqcup X'_+$ on $\mathbb{Z} + \frac{1}{2}$ defined in terms of $(P'_1, \dots, P'_D | Q'_1, \dots, Q'_D)$ as follows. If D is even, then we set

$$(4.12) \quad X'_+ = \left(P'_{D-1} + \frac{1}{2}, P'_{D-3} + \frac{1}{2}, \dots, P'_1 + \frac{1}{2} \right).$$

If D is odd, then we set

$$(4.13) \quad X'_- = \left(\frac{1}{2}, P'_{D-2} + \frac{1}{2}, P'_{D-4} + \frac{1}{2}, \dots, P'_1 + \frac{1}{2} \right).$$

(Observe that if D is odd, then $P'_D = 0$.) In both case (when D is either even or odd) we define X'_- as

$$(4.14) \quad X'_- = \left(-Q'_1 - \frac{1}{2}, -Q'_2 - \frac{1}{2}, \dots, -Q'_D - \frac{1}{2} \right).$$

Equations (4.12)-(4.14) define the embedding $\lambda \rightarrow X'$ of \mathbb{Y} into $\text{Conf}(\mathbb{Z} + \frac{1}{2})_0$. Under the embedding we get a point process on $\mathbb{Z} + \frac{1}{2}$. We will denote by the same symbol \underline{M} (as in Section 4.2) the point process obtained in this way from a probability measure M on \mathbb{Y} .

Theorem 4.2. (A) *The point process $\underline{M}_{z,z',\xi,\theta=\frac{1}{2}}$ is the Pfaffian L -ensemble in the sense of Section 2. The corresponding L -matrix is defined by equations (4.1)-(4.6) with*

$$(4.15) \quad h(x) = \begin{cases} \frac{[-2z+1]_{x-\frac{1}{2}}[-2z'+1]_{x-\frac{1}{2}}}{\Gamma(x+\frac{1}{2})} \xi^{\frac{x}{2}}, & x \in \mathbb{Z}_{\geq 0} + \frac{1}{2}, \\ \frac{[2z]_{-x-\frac{1}{2}}[2z']_{-x-\frac{1}{2}}}{\Gamma(-x+\frac{1}{2})} \xi^{-\frac{x}{2}}, & x \in \mathbb{Z}_{\leq 0} - \frac{1}{2}. \end{cases}$$

Moreover, we have

$$\text{Pf}(J + L) = (1 - \xi)^{-2zz'}.$$

(B) *The point process $\underline{M}_{\text{Pl},\eta,\theta=\frac{1}{2}}$ is the Pfaffian L -ensemble in the sense of Section 2. The corresponding L -matrix is the same as in Theorem 4.1: it is defined by equations (4.1)-(4.6) with the function h defined by equation (4.11).*

5. SPECIAL CLASS OF PFAFFIAN L -ENSEMBLES

In this Section we consider the Pfaffian L -ensemble on $\mathbb{Z} + \frac{1}{2}$ whose L -matrix is defined by equations (4.1)-(4.6).

Configurations $X \in \text{Conf}_0(\mathbb{Z} + \frac{1}{2})$ can be divided into two classes. The first class consists of configurations which do not include the point $\frac{1}{2}$. Such configurations have the form $X = X_- \sqcup X_+$, $X_+ = (x_1^+, x_2^+, \dots)$, $x_1^+ > \frac{1}{2}$. The second class consists of configurations that include the point $\frac{1}{2}$. For such configuration $X_+ = (\frac{1}{2}, x_1^+, x_2^+, \dots)$. For any $X \in \text{Conf}_0(\mathbb{Z} + \frac{1}{2})$ denote by \tilde{X} the configuration defined by

$$\begin{aligned} \tilde{X} &= \tilde{X}_- \sqcup \tilde{X}_+, \\ \tilde{X}_- &= X_-, \\ \tilde{X}_+ &= \begin{cases} (x_1^+ - 1, x_1^+, x_2^+ - 1, x_2^+, \dots), & X_+ \cap \frac{1}{2} = \emptyset, \\ (\frac{1}{2}, x_1^+ - 1, x_1^+, x_2^+ - 1, x_2^+, \dots), & X_+ \cap \frac{1}{2} \neq \emptyset. \end{cases} \end{aligned}$$

Definition 5.1. We say that $X \in \text{Conf}^L(\mathbb{Z} + \frac{1}{2})$ if

- $X \in \text{Conf}_0(\mathbb{Z} + \frac{1}{2})$
- $X_+ = (x_1^+ < x_2^+ < \dots)$

- all points of \tilde{X}_+ are different
- $|\tilde{X}_-| = |\tilde{X}_+|$
- $\tilde{X}_- = (x_1^- < x_2^- < \dots)$, where x_i^- has the same parity as i .

Let us introduce the following notation. Set $\prod(A; B) \equiv \prod_{i=1}^k \prod_{j=1}^l (a_i - b_j)$ for any two sets $A = (a_1, \dots, a_k)$, $B = (b_1, \dots, b_l)$, let $V(X)$ be the Vandermonde determinant associated with a set X ,

$$V(X) = \prod_{1 \leq i < j \leq N} (x_i - x_j), \quad X = (x_1, \dots, x_N),$$

and set $h(X) = \prod_{j=1}^N h(x_j)$. Definition (5.1) is justified by the following statement.

Proposition 5.2. *With L given by equations (4.1)-(4.6) we have*

$$(5.1) \quad \text{Pf } L(X|X) = \frac{V(\tilde{X}_-)V(\tilde{X}_+)}{\prod(\tilde{X}_+; \tilde{X}_-)} h(\tilde{X})$$

for $X \in \text{Conf}^L(\mathbb{Z} + \frac{1}{2})$ and 0 for all other $X \in \text{Conf}_0(\mathbb{Z} + \frac{1}{2})$.

Proof. This fact was first proved in Borodin and Strahov [15], Section 3.2. We reproduce here this proof (with minor changes) to make the paper self-contained.

The positive integer $d = |\tilde{X}_-| = |\tilde{X}_+|$ can be even or odd, depending on whether X includes the point $\mathfrak{x} = \frac{1}{2}$ or not. So we consider two cases.

Case 1. $X \cap \mathfrak{x} = \emptyset$

Given copies X', X'' of $X \in \mathfrak{X}$ in $\mathfrak{X}', \mathfrak{X}''$ we denote by $X' \uplus X''$ the set $(x'_1, x''_1, x'_2, x''_2, \dots)$. Then we have

$$\begin{aligned} \text{Pf } L(X|X) &= \text{Pf } L \left[X_- \sqcup X_+ | X_- \sqcup X_+ \right] \\ &= \text{Pf } L \left[(X'_- \uplus X''_-) \sqcup (X'_+ \uplus X''_+) | (X'_- \uplus X''_-) \sqcup (X'_+ \uplus X''_+) \right] \\ &= (-)^{\frac{d(d-1)}{2}} \cdot \text{Pf } L \left[X'_- | X'_- \right] \\ &\quad \times \text{Pf } L \left[X''_- \sqcup (X'_+ \uplus X''_+) | X''_- \sqcup (X'_+ \uplus X''_+) \right] \end{aligned}$$

as the function $L(x, y) = 0$ for any $x \in X'_-$ and any y which does not belong to X'_- (see equations (2.1)-(2.2) and (4.2)-(4.6)). We note that $L(x, y) = 0$, if $x, y \in \mathfrak{X}''_-$, or if $x, y \in X'_+ \uplus X''_+$. Therefore $|X''_-| = |X'_+| + |X''_+|$, or $|X''_-| = 2|X'_+|$, which means that $|\tilde{X}_+| = |\tilde{X}_-|$.

Consider $\text{Pf } L[X'_-|X'_-]$. Note that the matrix $L[X'_-|X'_-]$ is even dimensional, if $|X_-| = 2|X_+|$. Moreover the matrix $L[X'_-|X'_-]$ is the matrix whose (i, j) entry is, by definition, given by $\epsilon(x_i^-, x_j^-)$. Clearly, if x_1^- is even, the first row of this matrix consists of zeros only. Thus, if $\text{Pf } L[X'_-|X'_-] \neq 0$, x_1^- must be odd. Now assume that x_{2i-1}^- and x_{2i}^- have the same parity. In this case $(2i-1)^{\text{st}}$ and $2i^{\text{th}}$ rows of the matrix $L[X'_-|X'_-]$ are equal to each other. Therefore, if $\text{Pf } L[X'_-|X'_-] \neq 0$ the elements of the set $\tilde{X}_- = (x_1^-, x_2^-, \dots)$ are such that x_1^- is odd, x_2^- is even, x_3^- is odd and so on. This proves the condition on the parity for the configurations in $\text{Conf}^L(\mathbb{Z} + \frac{1}{2})$. Moreover, using the definition of Pfaffian it is not hard to conclude that $\text{Pf } L[X'_-|X'_-] = 1$ for the configurations with non-zero probabilities.

Since $|X_-''| = |X'_+| + |X''_+|$ the matrix $L[X_-'' \sqcup (X'_+ \uplus X''_+) | X_-'' \sqcup (X'_+ \uplus X''_+)]$ has the block structure:

$$\begin{bmatrix} \mathbb{O}_{d \times d} & Q_{d \times d} \\ -Q_{d \times d}^T & \mathbb{O}_{d \times d} \end{bmatrix}$$

with

$$Q_{d \times d} = \begin{bmatrix} \frac{h(x_1^-)h(x_1^+)}{x_1^- - x_1^+} & \frac{h(x_1^-)h({}^l x_1^+)}{x_1^- - {}^l x_1^+} & \cdots & \frac{h(x_1^-)h(x_{d/2}^+)}{x_1^- - x_{d/2}^+} & \frac{h(x_1^-)h({}^l x_{d/2}^+)}{x_1^- - {}^l x_{d/2}^+} \\ \vdots & & & & \\ \frac{h(x_d^-)h(x_1^+)}{x_d^- - x_1^+} & \frac{h(x_d^-)h({}^l x_1^+)}{x_d^- - {}^l x_1^+} & \cdots & \frac{h(x_d^-)h(x_{d/2}^+)}{x_d^- - x_{d/2}^+} & \frac{h(x_d^-)h({}^l x_{d/2}^+)}{x_d^- - {}^l x_{d/2}^+} \end{bmatrix},$$

where ${}^l x = x - 1$ (d is even). Thus we have

$$\begin{aligned} \text{Pf } L(X|X) &= (-)^{\frac{d(d-1)}{2}} \text{Pf} \begin{bmatrix} \mathbb{O}_{d \times d} & Q_{d \times d} \\ -Q_{d \times d}^T & \mathbb{O}_{d \times d} \end{bmatrix} = \det Q_{d \times d} \\ &= (-)^{\frac{d}{2}} (-)^{\frac{d(d-1)}{2}} \frac{V(\tilde{X}_-)V(\tilde{X}_+)}{\prod(\tilde{X}_+; \tilde{X}_-)} h(\tilde{X}) \end{aligned}$$

where we have used the formula for the Cauchy determinant. Noting that $(-)^{\frac{d(d-1)}{2} + \frac{d}{2}} = (-)^{\frac{d^2}{2}} = 1$ (as d is even) we obtain the formula stated in the Theorem.

Case 2. $X \cap \mathfrak{x} \neq 0$

The proof is very similar. We observe that any configuration X has a form

$$X = X_- \sqcup \mathfrak{x} \sqcup X_+$$

Then

$$\begin{aligned} \text{Pf } L(X|X) &= \text{Pf } L \left[X_- \sqcup \mathfrak{x} \sqcup X_+ | X_- \sqcup \mathfrak{x} \sqcup X_+ \right] = \\ &= \text{Pf } L \left[(X'_- \uplus X''_-) \sqcup (\mathfrak{x}', \mathfrak{x}'') \sqcup (X'_+ \uplus X''_+) | (X'_- \uplus X''_-) \sqcup (\mathfrak{x}', \mathfrak{x}'') \sqcup (X'_+ \uplus X''_+) \right] \\ &= (-)^{\frac{d(d-1)}{2}} \cdot \text{Pf } L \left[X'_-, \mathfrak{x}' | X'_-, \mathfrak{x}' \right] \\ &\quad \times \text{Pf } L \left[X''_- \sqcup \mathfrak{x}'' \sqcup (X'_+ \uplus X''_+) | X''_- \sqcup \mathfrak{x}'' \sqcup (X'_+ \uplus X''_+) \right] \end{aligned}$$

Clearly, $|X''_-| = |\mathfrak{x}' \sqcup (X'_+ \uplus X''_+)|$, otherwise $\text{Pf } L(X|X) = 0$. Thus \tilde{X}_- consists of odd number of elements, and $|\tilde{X}_-| = |\tilde{X}_+| = d$, d is odd, and we repeat the same computations as in the previous case. \square

Corollary 5.3. *Assume that the nonnegative function h in the definition of the L -matrix (see equations (4.1)-(4.6)) is chosen in such a way that*

$$\sum_{X: X \in \text{Conf}(\mathbb{Z} + \frac{1}{2})_0} \frac{V(\tilde{X}_-)V(\tilde{X}_+)}{\prod(\tilde{X}_+; \tilde{X}_-)} h(\tilde{X}) < \infty.$$

Then the L -matrix given by equations (4.1)-(4.6) defines a Pfaffian L -ensemble. Namely, we have

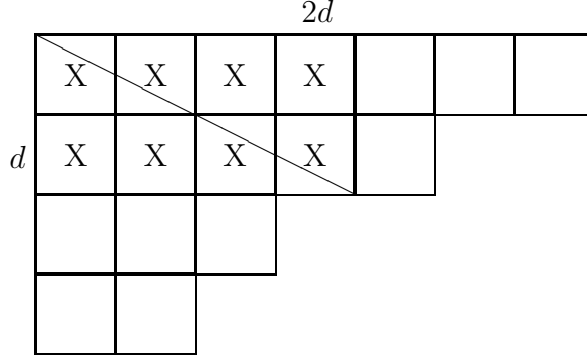
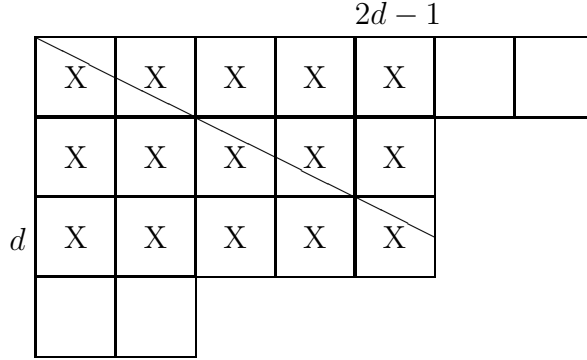
$$(5.2) \quad \text{Prob}_L(X) = \frac{1}{\text{Pf } (J + L)} \frac{V(\tilde{X}_-)V(\tilde{X}_+)}{\prod(\tilde{X}_+; \tilde{X}_-)} h(\tilde{X})$$

for $X \in \text{Conf}^L(\mathbb{Z} + \frac{1}{2})$ and 0 for all other $X \in \text{Conf}(\mathbb{Z} + \frac{1}{2})_0$.

6. THE PLANCHEREL MEASURES WITH THE JACK PARAMETERS

$$\theta = \frac{1}{2}, 2 \text{ AS } L\text{-ENSEMBLES}$$

6.1. Expression of the Plancherel measures with the Jack parameters $\theta = \frac{1}{2}, 2$ in terms of the Frobenius-type coordinates.

FIGURE 1. The box $(d, 2d)$ belongs to the Young diagram.FIGURE 2. The box $(d, 2d)$ does not belong to the Young diagram.

Proposition 6.1. *Let λ be a Young diagram, and let $(P_1, \dots, P_D | Q_1, \dots, Q_D)$ be the Frobenius of $\lambda \sqcup \lambda$ (see Section 4). We have*

$$\frac{1}{H(\lambda, \theta = 2)H'(\lambda, \theta = 2)} = \frac{\prod_{1 \leq i < j \leq D} (P_i - P_j)(Q_i - Q_j)}{\prod_{i=1}^D \prod_{j=1}^D (P_i + Q_j + 1) \prod_{i=1}^D P_i! Q_i!}.$$

Proof. In a given Young diagram λ we consider the diagonal $j = 2i$. There are two possible cases which are distinct from each other whether or not the box $(d, 2d)$ belongs to the Young diagram, see Figure 1 and Figure 2. The shape λ is divided into three pieces: the rectangular shape λ^\square of size $d \times 2d$ (in the first case shown on Figure 1), or of size $d \times (2d - 1)$ (in the second case shown on Figure 2); the diagram λ^+ formed by the boxes (ij) with $j \geq 2d$ (in the first case), or $j \geq 2d - 1$ (in the second case); and the diagram λ^- formed by the boxes (ij) with

$i > d$. Thus in both cases the diagram λ is composed in the following way

$$\lambda = \lambda^\square \sqcup \lambda^+ \sqcup \lambda^-.$$

In the subsequent calculations we exploit the following formulae

$$H(\lambda, \theta) = \prod_{i=1}^{l(\lambda)} \Gamma(\lambda_i - i\theta + l(\lambda)\theta + 1) \prod_{1 \leq i < j \leq l(\lambda)} \frac{\Gamma(\lambda_i - \lambda_j + (j-i)\theta + 1 - \theta)}{\Gamma(\lambda_i - \lambda_j + (j-i)\theta)},$$

$$H'(\lambda, \theta) = \prod_{i=1}^{l(\lambda)} \frac{\Gamma(\lambda_i - i\theta + l(\lambda)\theta + \theta)}{\Gamma(\theta)} \prod_{1 \leq i < j \leq l(\lambda)} \frac{\Gamma(\lambda_i - \lambda_j + (j-i)\theta)}{\Gamma(\lambda_i - \lambda_j + (j-i)\theta + \theta)}.$$

These expressions were obtained in Borodin and Olshanski [11], and hold true for any $\theta > 0$.

Consider first the case shown on Figure 1. For $1 \leq k \leq d$ introduce new coordinates

$$(6.1) \quad \lambda_k = p_k + 2k, \quad \lambda'_{2k-1} = \xi_k + k - 1, \quad \lambda'_{2k} = \eta_k + k.$$

In terms of these coordinates we obtain

$$(6.2) \quad \frac{1}{H(\lambda^+, \theta = 2)H'(\lambda^+, \theta = 2)} = \frac{\prod_{1 \leq k < m \leq d} (p_k - p_m)^2 ((p_k - p_m)^2 - 1)}{\prod_{k=1}^d p_k! (p_k + 1)!}.$$

Next we use the formulae stated in Proposition 3.1 to rewrite the expression

$$\frac{1}{H(\lambda^-, \theta = 2)H'(\lambda^-, \theta = 2)}$$

as follows

$$(6.3) \quad \frac{1}{H(\lambda^-, \theta = 2)H'(\lambda^-, \theta = 2)} = \frac{2^{-2|\lambda^-|} [\Gamma(\frac{1}{2})]^{2d}}{\prod_{k=1}^d \Gamma(\xi_k + \frac{1}{2}) \Gamma(\eta_k + 1) \Gamma(\xi_k) \Gamma(\eta_k + \frac{1}{2})}$$

$$\times \prod_{1 \leq k < m \leq d} (\eta_k - \eta_m)(\xi_k - \xi_m) \prod_{1 \leq 2k-1 < 2m \leq d} \left(\xi_k - \eta_m - \frac{1}{2} \right) \prod_{1 \leq 2k < 2m-1 \leq d} \left(\eta_k - \xi_m + \frac{1}{2} \right).$$

In addition, we find

$$(6.4) \quad H(\lambda^\square, \theta = 2) = \prod_{i=1}^d \prod_{k=1}^d (p_i + 2\xi_k)(p_i + 2\eta_k + 1),$$

and

$$(6.5) \quad H'(\lambda^\square, \theta = 2) = \prod_{i=1}^d \prod_{k=1}^d (p_i + 2\xi_k + 1)(p_i + 2\eta - k + 2).$$

The new coordinates introduced in equation (6.1) are related with the Frobenius coordinates of $\lambda \sqcup \lambda$ as

$$P = (p_1 + 1, p_1, \dots, p_d + 1, p_d), \quad Q = (2\xi_1 - 1, 2\eta_1, \dots, 2\xi_d - 1, 2\eta_d).$$

We rewrite equations (6.2)-(6.5) in terms of the Frobenius coordinates of $\lambda \sqcup \lambda$, and arrive to the formula stated in the statement of the Proposition. The second case (shown on Figure 2) can be considered in the same way. \square

Proposition 6.2. (A) *We have*

$$M_{P_1, \eta, \theta=2}(\lambda) = e^{-\eta(2\eta)^{\frac{1}{2} \sum_{i=1}^D (P_i + Q_i + 1)}} \frac{\prod_{1 \leq i < j \leq D} (P_i - P_j)(Q_i - Q_j)}{\prod_{i=1}^D \prod_{j=1}^D (P_i + Q_j + 1) \prod_{i=1}^D P_i! Q_i!}.$$

where $(P_1, \dots, P_D | Q_1, \dots, Q_D)$ are the Frobenius coordinates of $\lambda \sqcup \lambda$.
(B) *We have*

$$M_{P'_1, \eta, \theta=\frac{1}{2}}(\lambda) = e^{-\eta(2\eta)^{\frac{1}{2} \sum_{i=1}^D (P'_i + Q'_i + 1)}} \frac{\prod_{1 \leq i < j \leq D} (P'_i - P'_j)(Q'_i - Q'_j)}{\prod_{i=1}^D \prod_{j=1}^D (P'_i + Q'_j + 1) \prod_{i=1}^D P'_i! Q'_i!}.$$

where $(P'_1, \dots, P'_D | Q'_1, \dots, Q'_D)$ are the Frobenius coordinates of $\lambda' \sqcup \lambda'$.

Proof. These expressions follow from the formula in Proposition 6.1, and from equations (3.3)-(3.5). \square

6.2. Proof of Theorem 4.1 (B). Let λ be a Young diagram, and let $(P_1, \dots, P_D | Q_1, \dots, Q_D)$ be the Frobenius coordinates of $\lambda \sqcup \lambda$. It is not hard to check that if X is defined in terms of these Frobenius coordinates by equations (4.7)-(4.9), then $X \in \text{Conf}^L(\mathbb{Z} + \frac{1}{2})$ (see Definition 5.1). Conversely, for any $X \in \text{Conf}^L(\mathbb{Z} + \frac{1}{2})$ there exists a Young diagram λ , $\lambda \in \mathbb{Y}$, such that $X = X_- \sqcup X_+$ can be represented in terms of the Frobenius coordinates of $\lambda \sqcup \lambda$ as in equations (4.7)-(4.9). We conclude that there is a one-to-one correspondence between \mathbb{Y} and $\text{Conf}^L(\mathbb{Z} + \frac{1}{2})$, and this correspondence is defined by equations (4.7)-(4.9).

Consider the L -matrix defined by equations (4.1)-(4.6), with the weight function h defined by equation (4.11). Observe that if the condition $X \in \text{Conf}^L(\mathbb{Z} + \frac{1}{2})$ is not satisfied, then $\text{Pf}(X|X) = 0$. (This

follows from the very definition of $\text{Conf}^L(\mathbb{Z} + \frac{1}{2})$, see Definition 5.1). Therefore it is enough to show that

$$(6.6) \quad M_{\text{Pl}, \eta, \theta=2}(\lambda) = \frac{\text{Pf } L(X|X)}{\text{Pf}(J+L)},$$

where X is defined in terms of the Frobenius coordinates of $\lambda \sqcup \lambda$ as in equations (4.7)-(4.9). We use Proposition 6.2 (A), and rewrite $M_{\text{Pl}, \eta, \theta=2}(\lambda)$ in terms of the coordinates \tilde{X} as

$$M_{\text{Pl}, \eta, \theta=2}(\lambda) = e^{-\eta} \frac{V(\tilde{X}_-)V(\tilde{X}_+)}{\prod(\tilde{X}_+, \tilde{X}_-)} h(\tilde{X}).$$

(The coordinates \tilde{X} are constructed in terms of the coordinates X as it is explained in Section 5). Then Proposition 5.2 implies that

$$M_{\text{Pl}, \eta, \theta=2}(\lambda) = e^{-\eta} \text{Pf } L(X|X).$$

Since $M_{\text{Pl}, \eta, \theta=2}$ is a probability measure on \mathbb{Y} , we have

$$(6.7) \quad \begin{aligned} 1 &= \sum_{\lambda \in \mathbb{Y}} M_{\text{Pl}, \eta, \theta=2}(\lambda) \\ &= e^{-\eta} \sum_{X: X \in \text{Conf}^L(\mathbb{Z} + \frac{1}{2})} \text{Pf } L(X|X) \\ &= e^{-\eta} \sum_{X: X \in \text{Conf}(\mathbb{Z} + \frac{1}{2})_0} \text{Pf } L(X|X). \end{aligned}$$

(In the last equation we have used the fact that $\text{Pf } L(X|X) = 0$ for all $X \in \text{Conf}(\mathbb{Z} + \frac{1}{2})_0$ such that the condition $X \in \text{Conf}^L(\mathbb{Z} + \frac{1}{2})$ is not satisfied). This shows that

$$\sum_{X: X \in \text{Conf}(\mathbb{Z} + \frac{1}{2})_0} \text{Pf } L(X|X) = e^{\eta} < \infty.$$

Recall that $\text{Pf}(J+L)$ is defined as the sum $\sum_{X: X \in \text{Conf}(\mathbb{Z} + \frac{1}{2})_0} \text{Pf } L(X|X)$ provided that this sum is finite. Therefore $\text{Pf}(J+L) = e^{\eta}$, and formula (6.6) holds true. \square

6.3. Proof of Theorem 4.2 (B). Let λ be a Young diagram, and let $(P'_1, \dots, P'_D | Q'_1, \dots, Q'_D)$ be the Frobenius coordinates of $\lambda' \sqcup \lambda'$. Then equations (4.12)-(4.14) define a one-to-one correspondence between \mathbb{Y} and $\text{Conf}^L(\mathbb{Z} + \frac{1}{2})$. Define the L -matrix as in the proof of Theorem 4.1 (B) (i.e. by equations (4.1)-(4.6), with the weight function h defined by equation (4.11)). We need to show that

$$(6.8) \quad M_{\text{Pl}, \eta, \theta=\frac{1}{2}}(\lambda) = \frac{\text{Pf } L(X'|X')}{\text{Pf}(J+L)},$$

where X' is defined in terms of the Frobenius coordinates of $\lambda' \sqcup \lambda'$ as in equations (4.12)-(4.14). We use Proposition 6.2 (B), and rewrite $M_{\text{Pl}, \eta, \theta=\frac{1}{2}}$ as

$$M_{\text{Pl}, \eta, \theta=\frac{1}{2}}(\lambda) = e^{-\eta} \frac{V(\tilde{X}'_-)V(\tilde{X}'_+)}{\prod(\tilde{X}'_+, \tilde{X}'_-)} h(\tilde{X}'),$$

where the coordinates \tilde{X}' are related to the coordinates X' in the same way as the coordinates \tilde{X} are related to the coordinates X , see Section 5. Formula (6.8) is then obtained by the same argument as in the proof of Theorem 4.1 (B). \square

7. THE MIXED z -MEASURES WITH THE JACK PARAMETERS $\theta = \frac{1}{2}, 2$ AS L -ENSEMBLES

7.1. Expression of the z -measures with the Jack parameters $\theta = \frac{1}{2}, 2$ in terms of the Frobenius-type coordinates.

Proposition 7.1. (A) Let $\lambda \sqcup \lambda = (P_1, \dots, P_D | Q_1, \dots, Q_D)$ be the Frobenius notation for the Young diagram $\lambda \sqcup \lambda$, where D is the length of the diagonal in $\lambda \sqcup \lambda$, and P_i, Q_i are the Frobenius coordinates of $\lambda \sqcup \lambda$. The formula for the z -measure with the Jack parameter $\theta = 2$ (equation (3.2)) can be rewritten as follows

$$M_{z, z', \theta=2, \xi}(\lambda) = (1 - \xi)^{\frac{zz'}{2}} \xi^{\frac{1}{2} \sum_{i=1}^D (P_i + Q_i + 1)} \frac{\prod_{1 \leq i < j \leq D} (P_i - P_j)(Q_i - Q_j)}{\prod_{i=1}^D \prod_{j=1}^D (P_i + Q_j + 1)} \\ \times \prod_{i=1}^D \frac{[z + 1]_{P_i} [z' + 1]_{P_i} [-z]_{Q_i} [-z']_{Q_i}}{P_i! Q_i!}.$$

(B) Let $\lambda' \sqcup \lambda' = (P'_1, \dots, P'_D | Q'_1, \dots, Q'_D)$ be the Frobenius notation for the Young diagram $\lambda' \sqcup \lambda'$, where D is the length of the diagonal in $\lambda' \sqcup \lambda'$, and P'_i, Q'_i are the Frobenius coordinates of $\lambda' \sqcup \lambda'$. The formula for the z -measure with the Jack parameter $\theta = \frac{1}{2}$ can be rewritten as

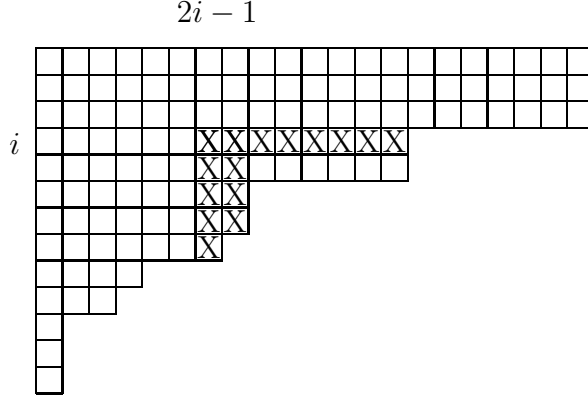


FIGURE 3. The distinguished shape associated with the box $(i, 2i - 1)$.

follows

$$M_{z, z', \theta = \frac{1}{2}, \xi}(\lambda) = (1 - \xi)^{2zz'} \xi^{\frac{1}{2} \sum_{i=1}^D (P'_i + Q'_i + 1)} \frac{\prod_{1 \leq i < j \leq D} (P'_i - P'_j)(Q'_i - Q'_j)}{\prod_{i=1}^D \prod_{j=1}^D (P'_i + Q'_j + 1)} \\ \times \prod_{i=1}^D \frac{[-2z + 1]_{P'_i} [-2z' + 1]_{P'_i} [2z]_{Q'_i} [2z']_{Q'_i}}{P'_i! Q'_i!}.$$

Proof. We start from the formula

$$M_{z, z', \theta = 2, \xi}(\lambda) = (1 - \xi)^{\frac{zz'}{2}} \xi^{|\lambda|} \frac{(z)_{\lambda, \theta=2} (z')_{\lambda, \theta=2}}{H(\lambda, \theta=2) H'(\lambda, \theta=2)}.$$

Given a box $(i, 2i - 1)$ of a Young diagram λ , consider the shape formed by the boxes

$$(i, 2i - 1), (i + 1, 2i - 1), \dots, (\lambda'_{2i-1}, 2i - 1); \\ (i, 2i), (i, 2i + 1), \dots, (i, \lambda_i);$$

and

$$(i + 1, 2i), (i + 2, 2i), \dots, (\lambda'_{2i}, 2i),$$

see Figure 3. The contribution of this shape to $(z)_{\lambda, \theta=2} (z')_{\lambda, \theta=2}$ is

$$(z)(z - 2) \dots (z - 2(\lambda'_{2i-1} - i))(z')(z' - 2) \dots (z' - 2(\lambda'_{2i-1} - i)) \\ \times (z + 1)(z + 2) \dots (z + (\lambda_i - 2i + 1))(z' + 1)(z' + 2) \dots (z' + (\lambda_i - 2i + 1)) \\ \times (z - 1)(z - 3) \dots (z + 1 - 2(\lambda'_{2i} - i))(z' - 1)(z' - 3) \dots (z' + 1 - 2(\lambda'_{2i} - i)).$$

This can be rewritten in terms of the Frobenius coordinates of $\lambda \sqcup \lambda$. Observe that the following relations hold true

$$P_{2i-1} = \lambda_i - 2i + 1, \quad Q_{2i-1} = 2\lambda'_{2i-1} - 2i + 1, \quad Q_{2i} = 2\lambda'_{2i} - 2i.$$

Using these relations we find

$$(z)_{\lambda; \theta=2} (z')_{\lambda; \theta=2} = \prod_{i=1}^D [z+1]_{P_i} [z'+1]_{P_i} [-z]_{Q_i} [-z']_{Q_i}.$$

Now we apply Proposition 6.1, and get the formula for $M_{z,z',\theta=2,\xi}(\lambda)$. The formula for $M_{z,z',\theta=\frac{1}{2},\xi}(\lambda)$ follows from the relation

$$M_{z,z',\theta=\frac{1}{2},\xi}(\lambda) = M_{-2z,-2z',\theta=2,\xi}(\lambda').$$

This relation is a simple consequence of Proposition 3.2. \square

7.2. Proof of Theorem 4.1 (A). We know (see the proof of Theorem 4.1 (B)) that equations (4.7)-(4.9) (expressing each $X \in \text{Conf}^L(\mathbb{Z} + \frac{1}{2})$ in terms of the Frobenius coordinates of $\lambda \sqcup \lambda$) define a one-to-one correspondence between \mathbb{Y} and $\text{Conf}^L(\mathbb{Z} + \frac{1}{2})$. Define the L -matrix by equations (4.1)-(4.6), with the weight function h given by equation (4.10). If the condition $X \in \text{Conf}^L(\mathbb{Z} + \frac{1}{2})$ is not satisfied, then $\text{Pf } L(X|X) = 0$. Therefore it is enough to show that

$$(7.1) \quad M_{z,z',\xi,\theta=2}(\lambda) = \frac{\text{Pf } L(X|X)}{\text{Pf}(J+L)},$$

where X is defined in terms of the Frobenius coordinates of $\lambda \sqcup \lambda$ as in equations (4.7)-(4.9). We use Proposition 7.1, and rewrite $M_{z,z',\xi,\theta=2}$ in terms of the coordinates X as

$$(7.2) \quad M_{z,z',\xi,\theta=2}(\lambda) = (1 - \xi)^{\frac{zz'}{2}} \frac{V(\tilde{X}_-) V(\tilde{X}_+)}{\prod(\tilde{X}_+, \tilde{X}_-)} h(\tilde{X}).$$

By the same argument as in the proof of Theorem 4.1 (B) we obtain equation (7.1) (with $\text{Pf}(J+L) = (1 - \xi)^{-\frac{zz'}{2}}$) from formula (7.2). \square

7.3. Proof of Theorem 4.2 (A). We use equations (4.12)-(4.14) (expressing each $X' \in \text{Conf}^L(\mathbb{Z} + \frac{1}{2})$ in terms of the Frobenius coordinates of $\lambda' \sqcup \lambda'$) to define a one-to-one correspondence between \mathbb{Y} and $\text{Conf}^L(\mathbb{Z} + \frac{1}{2})$. Observe that formula (7.1) can be rewritten in terms of the coordinates X' as

$$M_{z,z',\xi,\theta=\frac{1}{2}}(\lambda) = (1 - \xi)^{2zz'} \text{Pf } L(X'|X'),$$

where the L -matrix is defined as in the statement of Theorem 4.2 (A). This follows from Proposition 7.1 (B), Proposition 5.2, and equations

(4.12)-(4.14). By the same argument as in the proof of Theorem 4.2 (B) we obtain that $\text{Pf}(J + L) = (1 - \xi)^{-2zz'}$. Therefore,

$$M_{z,z',\xi,\theta=\frac{1}{2}}(\lambda) = \frac{\text{Pf } L(X'|X')}{\text{Pf}(J + L)},$$

i.e. $M_{z,z',\xi,\theta=\frac{1}{2}}$ defines a Pfaffian L -ensemble. \square

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